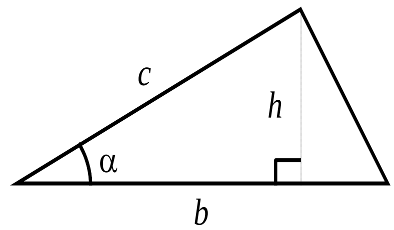
Applications of the Law of Sines

Area of a Triangle

The formula $A = \frac{1}{2}bh$ gives the area of a triangle.



In the diagram at the right, $\sin\alpha = \frac{h}{c}$, so $h = c\sin\alpha$. Using substitution, we derive the formula $A = \frac{1}{2}bc\sin\alpha$.

Depending on which angles and sides are known, the formulas $A = \frac{1}{2}ac\sin\beta$ and $A = \frac{1}{2}ab\sin\gamma$ can also be used.

Examples:

Find the area of the triangle with $\alpha = 39.4^\circ$, $b = 12.6$, and $c = 13.7$

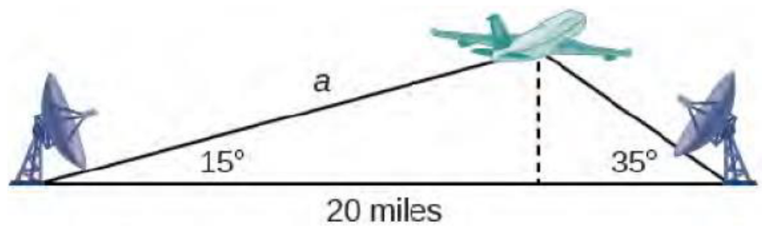
Find the area of a triangle with $\alpha = 56.3^\circ$, $\beta = 41.2^\circ$, and $a = 9.8$

Solving Applied Problems Using the Law of Sines

The more we study trigonometric applications, the more we discover that the applications are countless. Some are flat, diagram-type situations, but many applications in calculus, engineering and physics involve three dimensions and motion.

Example 7.2.2. Suppose two radar stations located 20 miles apart each detect an aircraft between them. The angle of elevation measured by the first station is 15 degrees, whereas the angle of elevation measured by the second station is 35 degrees. Find the altitude of the aircraft and round your answer to the nearest tenth of a mile.

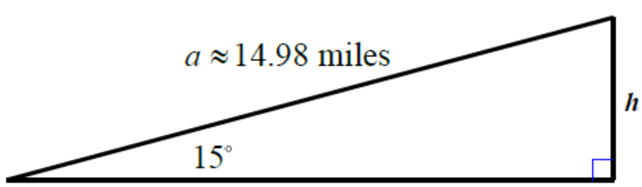
Solution. To find the altitude, or height, of the aircraft, we first sketch a triangle which reflects the information given to us in the problem. We then use the triangle to determine the distance from one station to the aircraft.



Letting a represent the distance from the first station to the aircraft, we look for an angle-side opposite pair from which we can determine the distance a. We know the measure of two angles in the triangle, but the measure of the angle opposite the side of length 20 miles is missing. Noting that the angles in a triangle add up to 180 degrees, we find the unknown angle measure to be $180^\circ - 15^\circ - 35^\circ = 130^\circ$. This gives us an angle-side opposite pair with known values and allows us to set up a Law of Sines relationship.

$$\frac{\sin(130^\circ)}{20} = \frac{\sin(35^\circ)}{a}$$
$$a\sin(130^\circ) = 20\sin(35^\circ)$$
$$a = \frac{20\sin(35^\circ)}{\sin(130^\circ)}$$
$$a \approx 14.98$$

The distance a, from the first station to the aircraft, is about 14.98 miles. Now that we know a, we can use right triangle relationships to solve for the height, h, of the aircraft.



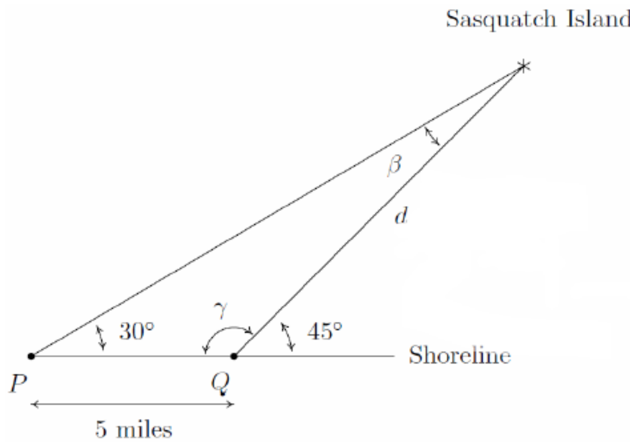
$$\sin(15^\circ) = \frac{h}{a}$$
$$h = a\sin(15^\circ)$$
$$h \approx 14.98\sin(15^\circ)$$
$$h \approx 3.88$$

The aircraft is at an altitude of approximately 3.9 miles.

□

Example 7.2.3. Sasquatch Island lies off the coast of Ippizuti Lake. Two sightings, taken 5 miles apart, are made to the island. The angle between the shore and the island at the first observation point is 30° and at the second point is 45°. Assuming a straight coastline, find the distance from the second observation to the island. What point on the shore is closest to the island? How far is the island from this point?

Solution. We sketch the problem below with the first observation point labeled as P and the second as Q.



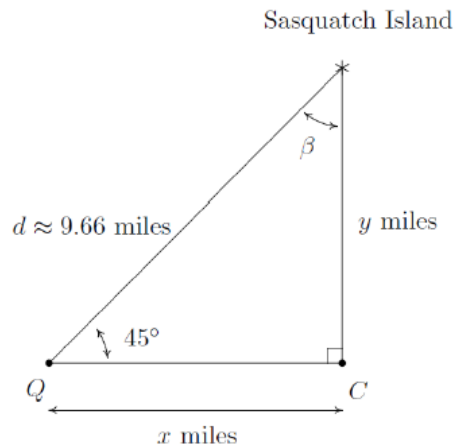
In order to use the Law of Sines to find the distance d from Q to the island, we first need to find the measure of β which is the angle opposite the side of length 5 miles. To that end, we note that the angles γ and 45° are supplemental, so that $\gamma = 180^\circ - 45^\circ = 135^\circ$. We can now find β.

$$\begin{aligned} \beta &= 180^\circ - 30^\circ - \gamma \\ &= 180^\circ - 30^\circ - 135^\circ \\ &= 15^\circ \end{aligned}$$

By the Law of Sines, we have

$$\begin{aligned} \frac{d}{\sin(30^\circ)} &= \frac{5}{\sin(15^\circ)} \\ d &= \frac{5\sin(30^\circ)}{\sin(15^\circ)} \\ d &\approx 9.66 \text{ miles} \end{aligned}$$

Next, to find the point on the coast closest to the island, which we've labeled as C, we need to find the perpendicular distance from the island to the coast.[1]



Let x denote the distance from the second observation point Q to the point C and let y denote the distance from C to the island. Using the right triangle definition of sine, we get

$$\sin(45^\circ) = \frac{y}{d}$$
$$y = d\sin(45^\circ)$$
$$y \approx 9.66\left(\frac{\sqrt{2}}{2}\right)$$
$$y \approx 6.83 \text{ miles}$$

Hence, the island is approximately 6.83 miles from the coast. To find the distance from Q to C, we note that $\beta = 180^\circ - 90^\circ - 45^\circ = 45^\circ$ so by symmetry, we get $x = y \approx 6.83$ miles. Hence, the point on the shore closest to the island is approximately 6.83 miles down the coast from the second observation point.

□

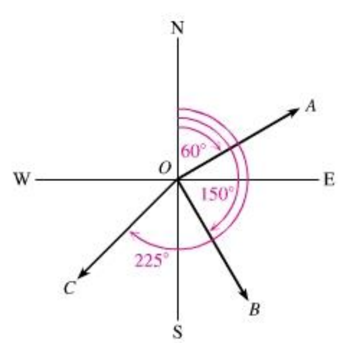
We close this section with the encouragement that, by working through the many problems in the Exercises, you will become proficient in applying the Law of Sines to real-world applications, and will be ready to move on to the Law of Cosines in **Section 7.3**.

[1] Do you see why C must lie to the right of Q?

Bearing: The measure of an angle that describes the direction of a ray is called the bearing. Bearing is the clockwise angle from due north.

Another way to express bearing is to describe the acute angle that the ray makes with a ray pointing due north or south. For example:

N60°E is a bearing of 60° east of north
S30°E is a bearing of 30° east of south
S45°W is a bearing of 45° west of south



Example: During an important NATO exercise, an F-14 Tomcat left the carrier Nimitz on a course with a bearing of 34° and flew 400 miles. Then the F-14 flew for some distance on a course with a bearing of 162°. Finally, the plane flew back to its starting point on a course with a bearing of 308°. What distance did the plane fly on the final leg of the journey? Round to the nearest tenth of a mile.